SECOND MIDDLE EUROPEAN MATHEMATICAL OLYMPIAD

OLOMOUC
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Team Problems and Solutions

T–1 Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$xf(x + xy) = xf(x) + f(x^2)f(y)$$

for all $x, y \in \mathbb{R}$.

Solution. Setting $x = y = 0$ in

$$xf(x + xy) = xf(x) + f(x^2)f(y) \tag{0}$$

we get $f(0) = 0$. Using this in (0) with $y = -1$ we obtain

$$xf(x) + f(x^2)f(-1) = 0. \tag{1}$$

Let us distinguish the cases $f(-1) = 0$ and $f(-1) \neq 0$.

The case $f(-1) = 0$. It follows from (1) that $f(x) = 0$ for all $x \neq 0$. As we already know, $f(0) = 0$. Thus we get the zero function $f(x) = 0$, which is obviously a solution.

The case $f(-1) \neq 0$. Setting $x = -1$ in (1) yields $f(1) = 1$. Using this in (1) with $x = 1$, we get $f(-1) = -1$ and hence (1) can be transformed to

$$xf(x) = f(x^2). \tag{2}$$

Put now $y = x - 1$ in (0) to get

$$xf(x^2) = xf(x) + f(x^2)f(x-1). \tag{3}$$

Summing up (2) and (3) we obtain the equation

$$f(x^2)(f(x-1) - (x-1)) = 0. \tag{4}$$

Assume that $f(a) = 0$ for some $a \neq 0$. Then $f(a^2) = 0$ by (2) and hence (0) with $x = a$ implies that $af(a + ay) = 0$, i.e. $f(a + ay) = 0$. Since y is arbitrary here, we get $f(-1) = 0$, which is not the case. Therefore, for any $x \neq 0$ we have $f(x) \neq 0$, and hence $f(x^2) \neq 0$ as well. Thus (4) leads to the conclusion that $f(x-1) = x - 1$ for any $x \neq 0$, i.e. $f(x) = x$ for any $x \neq -1$. Since we already know that $f(-1) = -1$, we get the identity function $f(x) = x$, which is obviously a solution.

T–2 Let $n \geqslant 2$ be an integer. There are n positive integers written on a blackboard. In each step we choose two of the numbers on the blackboard and replace each of them by their sum. Determine all values of n for which it is always possible to get n identical integers in a finite number of steps.

Solution. Starting from the n-tuple $(2, 2, 1, 1, \ldots, 1)$ with any $n \geqslant 3$, we get always an n-tuple in which the number of maximal values is *even*. Hence no odd $n \geqslant 3$ is as required.

Let us show by induction that any even $n \geqslant 2$ is satisfactory, which is obvious if $n = 2$. For an even $n \geqslant 4$, by the induction hypothesis, we can transform any initial n-tuple to $(a, a, \ldots, a, b, b)$. If $a \neq b$, we apply repeatedly some of the following series of steps, which always lead to an n-tuple of type $(\underbrace{a, \ldots, a}_{k}, \underbrace{b, \ldots, b}_{n-k})$ in which the number k may differ from the initial value $k = n - 2$ (remaining to be *even*):

$$\text{series } \alpha\text{:} \quad (\underbrace{a, \ldots, a}_{k}, \underbrace{b, \ldots, b}_{n-k}) \to (\underbrace{2a, \ldots, 2a}_{k}, \underbrace{b, \ldots, b}_{n-k}),$$

$$\text{series } \beta\text{:} \quad (\underbrace{a, \ldots, a}_{k}, \underbrace{b, \ldots, b}_{n-k}) \to (\underbrace{a, \ldots, a}_{k}, \underbrace{2b, \ldots, 2b}_{n-k}),$$

$$\text{series } \gamma_1 \text{ (if } k \leqslant n-k\text{):} \quad (\underbrace{a, \ldots, a}_{k}, \underbrace{b, \ldots, b}_{n-k}) \to (\underbrace{a+b, \ldots, a+b}_{2k}, \underbrace{b, \ldots, b}_{n-2k}),$$

$$\text{series } \gamma_2 \text{ (if } k \geqslant n-k\text{):} \quad (\underbrace{a, \ldots, a}_{k}, \underbrace{b, \ldots, b}_{n-k}) \to (\underbrace{a, \ldots, a}_{2k-n}, \underbrace{a+b, \ldots, a+b}_{2(n-k)}).$$

To describe our procedure, we introduce the notation $c = 2^{P(c)}N(c)$ for any positive integer c, where $P(c) \geqslant 0$ and $N(c)$ is odd. To each n-tuple $(\underbrace{a, \ldots, a}_{k}, \underbrace{b, \ldots, b}_{n-k})$ with $a \neq b$, let us apply

▷ series α if $P(a) < P(b)$,
▷ series β if $P(a) > P(b)$,
▷ series γ_1 or γ_2 if $P(a) = P(b)$ (and hence $N(a) \neq N(b)$).

Using the series α and β, the numbers $N(a)$, $N(b)$ do not change, while the series γ_1 and γ_2 cause the changes exactly one of them, namely

$$N(b) \to \frac{N(a)+N(b)}{2^m}, \quad \text{or} \quad N(b) \to \frac{N(a)+N(b)}{2^m} \quad \text{respectively,}$$

where $m = P(N(a)+N(b)) \geqslant 1$ and hence

$$\frac{N(a)+N(b)}{2^m} \leqslant \frac{N(a)+N(b)}{2} < \max(N(a), N(b))$$

(recall that $N(a) \neq N(b)$). Consequently, throughout our procedure, the value of $\max(N(a), N(b))$ is a nonincreasing variable, and hence constant after a finite numbers of series. From this moment, we must still have either $N(a) \geqslant N(b)$, or $N(a) \leqslant N(b)$. This excludes either series γ_1, or series γ_2 from future applications, in which, therefore, all possible changes of the parameter k are either $k \to 2k$, or $(n-k) \to 2(n-k)$. Since this can repeat only r times, where $2^r \leqslant n$, at the end we always we get an n-tuple $(a, \dots, a, b, \dots, b)$ for which (if $a \neq b$) the continuation of our procedure reduces only to the series α and β. Applying now either α, or β exactly $|P(a)-P(b)|$ times, we get an n-tuple $(a', \dots, a', b', \dots, b')$ with $P(a') = P(b')$. Since γ_1, γ_2 are already excluded, we have $a' = b'$, which completes the induction proof.

Another solution (German team, adapted). We show *without* induction on n that any even $n = 2k$ is satisfactory. At the beginning in the initial $2k$-tuple $(a_1, \dots, a_{2k})$ we replace every pair (a_{2i-1}, a_{2i}) (for $i = 1, \dots, k$) by the pair $(a_{2i-1} + a_{2i}, a_{2i-1} + a_{2i})$. From now on, we shall have always identical numbers on the $(2i-1)$th and $(2i)$th position. Hence because of brevity we shall work with k-tuples $(x, y, z, \dots)$ instead of $2k$-tuples $(x, x, y, y, z, z, \dots)$. We are allowed to do the following transformations on the k-tuples:

▷ choose two of the numbers x, y and replace each of them by their sum (this corresponds with two steps $(\dots, x, x, \dots, y, y, \dots) \to (\dots, x+y, x, \dots, x+y, y, \dots) \to (\dots, x+y, x+y, \dots, x+y, x+y, \dots)$ performed on the $2k$-tuple);
▷ choose one number x and multiply it by 2 (this corresponds with one step $(\dots, x, x, \dots) \to (\dots, x+x, x+x, \dots)$;
▷ divide all numbers by 2 (this obviously does not affect anything; formally we could remember how many times we have performed this dividing and multiply all the numbers by the proper power of two at the end).

Our aim is to obtain k identical numbers. We reach it by iterating the following algorithm:

1. While there are at least two odd numbers, find the minimum and the maximum odd number and replace each of them by their (even) sum.
2. If there is one odd number left after finishing the first step, multiply it by two.
3. Divide all numbers by 2.

Clearly, after each iteration, the maximum number among all k numbers either decreases or does not change. As this maximum is permanently a positive integer, after a finite number of iterations, it fixes at the value M and does not change anymore. From now on, look at the number N of M's in the k-tuple.

Obviously M is odd (otherwise it would decrease in the third step in the next iteration). If $N < k$, then there is at least one number m with $m < M$. If m is odd, after the next iteration N decreases. As it is impossible to increase N in the iterations, it must be constant after a finite number of steps and there must be only even m with $m < M$. But every even m is divided by 2 in each iteration and after some iterations some odd number less than M must appear. So there are no numbers less than M, which completes the proof.

T–3 An acute-angled triangle ABC is given. Let E be a point such that B and E lie on different sides of the line AC, and let D be an interior point of the segment AE. Suppose that $\angle ADB = \angle CDE$, $\angle BAD = \angle ECD$ and $\angle ACB = \angle EBA$. Prove that B, C and E are collinear.

Solution. Condition $\angle ADB = \angle CDE$ motivates us to reflect B over AE to B' (Fig. 3). Then C, D and B' are collinear and $\angle EAB' = \angle EAB = \angle ECD = \angle ECB'$, so $B'ACE$ is a cyclic quadrilateral. This implies that $\angle ECA = \pi - \angle EB'A = \pi - \angle EBA = \pi - \angle ACB$, hence $\angle ECA + \angle ACB = \pi$ and thus B, C, E are collinear.

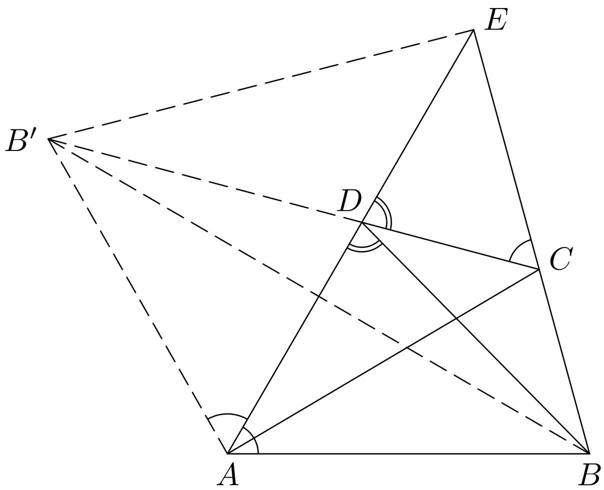

Fig. 3

Comment. With the same success, we can reflect C over AE to get a point C' collinear with B, D and such that $ABEC'$ is cyclic, hence $\angle ECA = \angle EC'A = \pi - \angle EBA = \pi - \angle ACB$, i.e. $\angle ECA + \angle ACB = \pi$ again.

Because of the proved collinearity of B, C, E, the condition $\angle ACB = \angle EBA$ implies that $AB = AC$, while the condition $\angle BAD = \angle ECD$ implies that $ABCD$ must be a cyclic quadrilateral. The last fact serves as a good motivation for another solution. Before we present it, let us note that the situations described in the statement of the problem *do exist* and all of them are of the following form:

ABC is an isosceles triangle with $AB = AC$, points B, C, E are collinear (C is between B and E) and AE cuts the circumcircle of ABC at D.

Another solution. Suppose that B, C, E are not collinear. The line through B, which is parallel to CE, meets the lines CD and AD at C' and E', respectively. Since $\angle E'C'D = \angle ECD = \angle BAD$, the quadrilateral $ABC'D$ is cyclic (Fig. 4). Denote its circumcircle by $\mathcal{K}$. We have $\angle AC'B = \angle ADB = \angle CDE = \angle C'DE = \angle ABC'$, i.e. $\angle AC'B = \angle ABC'$ (so ABC' is an isosceles triangle).

Suppose that C lies inside the segment $C'D$. Then C lies inside $\mathcal{K}$ (on the same side of the line AB as C'), therefore $\angle ACB > \angle AC'B = \angle ABC' = \angle ABE' > \angle ABE$ (because E lies between A and E'), which contradicts to $\angle ACB = \angle EBA$.

Similarly, if C does not lie on the segment $C'D$, then C lies outside $\mathcal{K}$ (on the same side of the line AB as C'), therefore $\angle ACB < \angle AC'B = \angle ABC' = \angle ABE' < \angle ABE$ (because E' lies between A and E), which again contradicts to $\angle ACB = \angle EBA$.

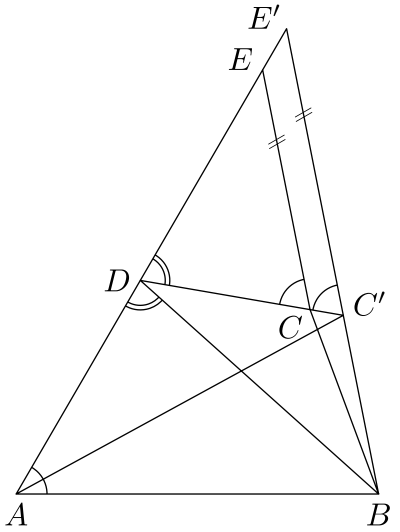

Fig. 4

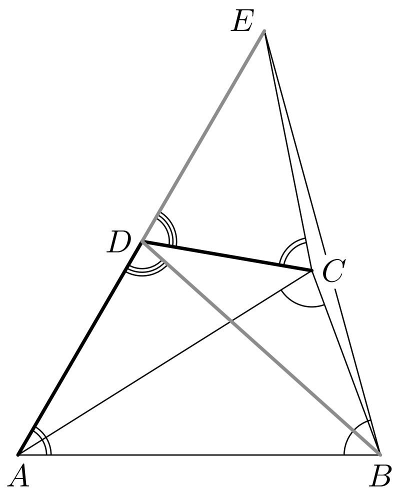

Fig. 5

Another solution (Karel Horák, Czech Republic). From the given equalities of angles it follows that triangles ABD and CED are similar (Fig. 5). From that similarity we immediately get that triangles ACD and BED are similar (by *sas*, same angles at common vertex D, and proportional sides). From the equal angles BED and ACD if follows that the sum of three angles BCA, ACD and DCE is equal to the sum of angles in the triangle ABE, so E, C, and B are collinear.

T–4 Let n be a positive integer. Prove that if the sum of all positive divisors of n is a perfect power of 2, then the number of these divisors is also a perfect power of 2.

Solution. Suppose that $n = p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$, where $p_1, \dots, p_k$ are distinct primes and $s_i \geqslant 1$ for each i, and that the sum of all positive divisors of n, which is given by

$$(1 + p_1 + p_1^2 + \dots + p_1^{s_1})(1 + p_2 + p_2^2 + \dots + p_2^{s_2}) \dots (1 + p_k + p_k^2 + \dots + p_k^{s_k}),$$

is a perfect power of 2. Then each of the factors

$$f_i = 1 + p_i + p_i^2 + \dots + p_i^{s_i}$$

is also a perfect power of 2 greater than 1 and hence both p_i and s_i are odd. Suppose that $s_i > 1$. In this case we have

$$f_i = (1 + p_i)(1 + p_i^2 + p_i^4 + \dots + p_i^{s_i - 1}).$$

Since f_i has no odd divisor greater than 1, the even integer $s_i - 1$ (which is supposed to be positive) must be of the form $4k+2$ and thus we can make another factorization

$$f_i = (1 + p_i)(1 + p_i^2)(1 + p_i^4 + p_i^8 + \cdots + p_i^{s_i-3}).$$

Consequently, both $1 + p_i$ and $1 + p_i^2$ are powers of 2, hence $1 + p_i \mid 1 + p_i^2$, which contradicts to $1 + p_i^2 = (1 + p_i)(p_i - 1) + 2$ (as $1 + p_i \mid 2$ is impossible). This means that $s_i = 1$ for each i and thus the number of divisors of n equals 2^k.

Note that the above solution can be finished without observing the fact that $1+p_i$ and $1 + p_i^2$ cannot be powers of 2 at the same time. Indeed, repeating the procedure of factorization we get finally

$$f_i = (1 + p_i)\big(1 + p_i^2\big)\big(1 + p_i^4\big) \ldots \Big(1 + p_i^{2^{t_i}}\Big),$$

hence $s_i = 2^{t_i+1} - 1$ with some $t_i \geqslant 0$ for each i and thus the number of divisors of n equals $2^{k+t_1+t_2+\cdots+t_k}$. (As we know from the original solution, $t_i = 0$ for each i.)