EUROPEAN
MIDDLE MATHEMATICAL
SECOND OLYMPIAD
OLOMOUC
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Individual Problems and Solutions

I–1 Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive integers such that $a_n < a_{n+1}$ for all $n \geqslant 1$. Suppose that for all quadruples of indices (i, j, k, l) such that $1 \leqslant i < j \leqslant k < l$ and $i+l = j+k$, the inequality $a_i + a_l > a_j + a_k$ is satisfied. Determine the least possible value of a_{2008}.

Solution (Jaromír Šimša, Czech Republic). Since $a_2 - a_1 \geqslant 1$ and $a_{n+2} - a_{n+1} \geqslant (a_{n+1} - a_n) + 1$ (by applying the quadruple $(n, n+1, n+1, n+2)$ for each n), induction yields $a_{n+1} - a_n \geqslant n$ for all $n \geqslant 1$. Thus $a_{n+1} \geqslant n + a_n$ (and $a_1 \geqslant 1$), hence induction again yields $a_n \geqslant \frac{1}{2}(n^2 - n + 2)$. Since the sequence $a_n = \frac{1}{2}(n^2 - n + 2)$ is as required (transform $a_i + a_l > a_j + a_k$ to $i^2 + l^2 > j^2 + k^2$ and substitute $i = d - y$, $l = d + y$, $j = d - x$, $k = d + x$, where $0 \leqslant x < y$), the smallest value of a_{2008} is 2,015,029.

I–2 Consider a chessboard $n \times n$ where $n > 1$ is a positive integer. We select the centers of $2n - 2$ squares. How many selections are there such that no two selected centers lie on a line parallel to one of the diagonals of the chessboard?

Solution. By a k-diagonal we mean any chessboard diagonal formed by k squares, where $1 \leqslant k \leqslant n$. Since the number of stones is $2n-2$, while the number of chessboard diagonals in one direction is $2n-1$ and two of them, which are 1-diagonals, must not be occupied by stones simultaneously, we can conclude that each k-diagonal with $k > 1$ contains exactly 1 stone and that exactly two of the 4 corner squares (1-diagonals) are occupied (and lie on the same border side). Let us call two different directions of diagonals as A and B.

Now let us consider the set P of all the pairs (s, f), for which the stone s lies on the same diagonal as the unoccupied ("free") square f. There are exactly $n^2 - 2n + 2$ free squares on the chessboard, two of them are corner, hence for each of the $n^2 - 2n$ free squares f which lie on two k-diagonals with $k > 1$, we have $(s, f) \in P$ for exactly two stones s. Thus the total number p of the pairs in P is given by the formula

$$p = 2(n^2 - 2n) + 2 = 2n^2 - 4n + 2,$$

where $+2$ stands for the two free corner squares.

If a stone s lies on the intersection of a k_1-diagonal and a k_2-diagonal with $k_1, k_2 > 1$, then the number of pairs $(s, f) \in P$ with this s equals $k_1 + k_2 - 2$. The same holds also for the two other stones with $\{k_1, k_2\} = \{1, n\}$. Obviously, for any stone we have $k_1 + k_2 \geqslant n + 1$ with equality iff the stone lies on a border square. Thus for each stone s, the number of pairs $(s, f) \in P$ is at least $n - 1$, and therefore

$$p \geqslant (2n - 2)(n - 1) = 2n^2 - 4n + 2.$$

Since we have the equality, *all the stones must lie on the boarder squares of the chessboard.*

If we put some stones (even no stone) on the first horizontal row in any way, then the border squares for the other stones are determined in exactly one way. To see this, consider separately the four corner squares and then, for each k, $1 < k < n$, the pair of k-diagonals one direction together with the pair of $(n + 1 - k)$-diagonals in the other direction. Hence there are exactly 2^n possibilities how to distribute the stones on the chessboard as required.

Comment. The proof of the fact that all the stones must lie on some of the border squares from the preceding solution can be presented in the following algebraic form without counting the pairs (s, f).

Consider the chessboard $n \times n$ as the grid $\{0, 1, \ldots, m\} \times \{0, 1, \ldots, m\}$ with $m = n - 1$, in which the occupied squares are represented by points (a_i, b_i) with $i = 1, 2, \ldots, 2m$. Since $a_i - b_i$ are $2m$ distinct integers from $\{-m, -m+1, \ldots, m-1, m\}$ and the boundary values $\pm m$ are not reached simultaneously, the values of $|a_i - b_i|$ (in nondecreasing order) are the numbers

$$0, 1, 1, 2, 2, \ldots, m - 1, m - 1, m$$

whose sum equals m^2. Thus we have

$$\sum_{i=1}^{i=2m} |a_i - b_i| = m^2 \tag{1}$$

and, similarly,

$$\sum_{i=1}^{i=2m} |a_i + b_i - m| = m^2. \tag{2}$$

Summing (1) and (2) and taking into account the identity

$$|a-b|+|a+b-m| = \max(|2a-m|,|2b-m|) \quad \text{for any } a,b,m \in \mathbb{R},$$

we obtain the equality

$$\sum_{i=1}^{i=2m} \max(|2a_i-m|,|2b_i-m|) = 2m^2. \tag{3}$$

Since $|2a_i-m| \leqslant m$ and $|2b_i-m| \leqslant m$ for each i, the following inequality

$$\sum_{i=1}^{i=2m} \max(|2a_i-m|,|2b_i-m|) \leqslant 2m \cdot m = 2m^2$$

holds and hence the equality (3) implies that

$$\max(|2a_i-m|,|2b_i-m|) = m \quad \text{for any } i=1,2,\dots,2m.$$

This means that any (a_i,b_i) is a boundary point of the grid and the proof is complete.

Another solution (Bernd Mulansky, Germany). The first paragraph is identical with that from the first solution. It follows that each satisfactory distribution of the $2n-2$ stones can be derived as a result of the following procedure in n steps:

- ▷ Step 1: One stone is placed on one of the two 1-diagonals of direction A.
- ▷ Step k (where $2 \leqslant k \leqslant n-1$): Two stones are placed, each on one of the two k-diagonals of direction A.
- ▷ Step n: One stone is placed on the n-diagonal of direction A.

Notice that for each $m=1,2,\dots,n-1$ the following conclusion clearly holds: after m steps of our procedure, well-done in the sense that no two stones were placed on the same diagonal (of direction B), all the $2m-1$ longest k-diagonals of direction B (those with $k \geqslant n+1-m$) are occupied by stones. Consequently, if in addition $m < n-1$, in the next step $m+1$ the two stones must be placed on the border squares of the two $(m+1)$-diagonals of direction A (their other squares lie on the occupied diagonals of direction B) and there are exactly two ways in which this can be well-done. Analogously for the case $m+1=n$. Thus we have two possibilities in each of the n steps and the number of all satisfactory distributions equals 2^n.

Another solution (Pavol Novotný, Slovakia). Let us colour the chessboard squares as usual, with the black square in the left upper corner. It is easy to show that $n-1$ stones must be placed on the black squares (let us call them black stones), analogously for the $n-1$ white stones. The number s_n of all satisfactory stone distributions on the chessboard $n \times n$ is equal to the product $b_n \cdot w_n$, where b_n and w_n are the numbers of satisfactory distributions of black and white stones, respectively. We have $w_1=1$, $w_2=w_3=2$, $b_1=1$, $b_2=2$ and $b_3=4$. Easy arguments show that for each $n \geqslant 3$, $w_n = 2w_{n-2}$,[1] and $b_n = 2w_{n-1}$,[2] hence $s_n = b_n w_n = 4w_{n-2}w_{n-1} = 2b_{n-1}w_{n-1} = 2s_{n-1}$ and the result $s_n = 2^n$ follows.

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[1] Remove two white 2-diagonals of one direction and two white $(n-1)$-diagonals of the other direction; the remaining white squares form the same diagonals as white squares of the chessboard $(n-2) \times (n-2)$.

[2] Remove one black n-diagonal; the remaining black squares form the same diagonals as the *white* squares of the chessboard $(n-1) \times (n-1)$.

I–3 Let ABC be an isosceles triangle with $|AC| = |BC|$. Its incircle touches AB and BC at D and E, respectively. A line (different from AE) passes through A and intersects the incircle at F and G. The lines EF and EG intersect the line AB at K and L, respectively. Prove that $|DK| = |DL|$.

Solution. In view of symmetry, suppose that $AF < AG$, and, in addition, that G is on the smaller arc DE (for the other case see the last two sentences below).

If the incircle touches AC at J, then $\angle CAB = \angle CJE = \angle JDE = \angle JFE$ (Fig. 1), hence $AJFK$ is a cyclic quadrilateral. Thus $\angle AJK = \angle AFK = \angle EFG = \angle LEB$, which implies that AJK and BEL are congruent triangles. Since K and L are inner points of the segment AB, $AK = BL$ means that $DK = DL$.

If G is on the larger arc DE (between E and J), then K, A, B, L is the order of these collinear points and the cyclic quadrilateral is $AKJF$. The rest of the proof is the same.

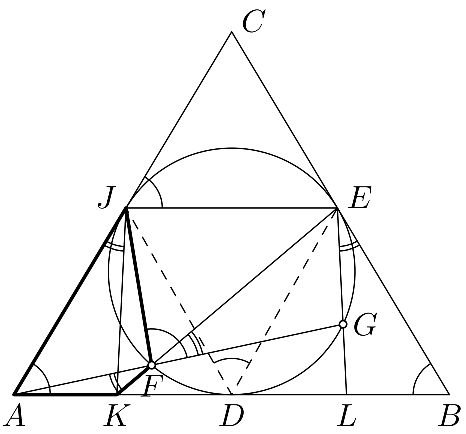

Fig. 1

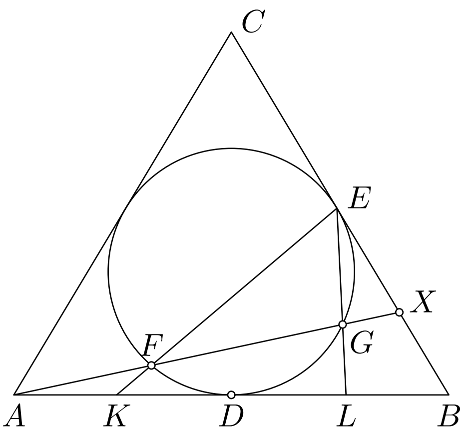

Fig. 2

Another solution (Tomáš Pavlík, Czech Republic). Let us denote X the intersection of line AF with side BC of the given triangle (Fig. 2). The power of the point X with regard to the incircle of ABC gives $XE^2 = XF \cdot XG$ which means that

$$\frac{XG}{XE} = \frac{XE}{XF}. \tag{1}$$

Let us write Menelaos theorem for triangle ABX and lines EG and EF, respectively:

$$\frac{AL}{LB} \cdot \frac{BE}{EX} \cdot \frac{XG}{GA} = 1 \quad \text{and} \quad \frac{AK}{KB} \cdot \frac{BE}{EX} \cdot \frac{XF}{FA} = 1.$$

With help of (1) we can rewrite both the last equalities as

$$\frac{XE}{XF} \cdot \frac{AL \cdot BE}{LB \cdot GA} = 1 \quad \text{and} \quad \frac{XE}{XF} \cdot \frac{KB \cdot FA}{AK \cdot BE} = 1$$

or

$$\frac{AL \cdot BE}{LB \cdot GA} = \frac{KB \cdot FA}{AK \cdot BE}$$

which gives

$$\frac{AK \cdot AL \cdot BE^2}{KB \cdot LB \cdot FA \cdot GA} = 1,$$

hence

$$AK \cdot AL = KB \cdot LB$$

as $AF \cdot AG = AD^2 = BD^2 = BE^2$ clearly holds.

Depending on the position of point G, the points K and L lie inside or outside the segment AB simultaneously, according to that we choose plus or minus sign in

$$AK \cdot (AB \pm BL) = AK \cdot AL = KB \cdot LB = (AB \pm AK) \cdot BL$$

which results into $AK = BL$ in both cases. This is equivalent to the wanted equality $DK = DL$.

I–4 Find all integers k such that for every integer n, the numbers $4n+1$ and $kn+1$ are relatively prime.

Solution. Since $4n+1$ is odd, the identity $k-4 = k(4n+1) - 4(kn+1)$ shows that $4n+1$ and $kn+1$ are relatively prime if $k-4$ has not any odd divisor $p > 1$, i.e. if $k - 4 = \pm 2^k$ with any nonnegative integer k.

On the other hand, if $k-4$ has got an odd divisor $p > 1$, then we can easily find a multiple of p of the form $4n+1$ (for example, the number p^2 or simply one of the numbers p, $3p$). For any number $4n+1$ being a multiple of p, the above identity implies that $p \mid kn+1$, hence $4n+1$ and $kn+1$ are not relatively prime.

Answer: $k = 4 \pm 2^k$, where $k = 0, 1, 2, \ldots$