Solutions for the problems of the Autumn round of the 30th Tournament of Towns

A-level, Juniors

1. Clearly, the western half of the board has 50 Queens, as does the northern half of the board. Assume by symmetry that the northwestern quadrant is empty. Then 50 Queens must be in the southwestern quadrant and another 50 in the northeastern quadrant. Hence the southeastern quadrant is also empty. However, the squares in the southwestern and the northeastern quadrants all lie on 99 diagonals going between the southwest and the northeast. By the Pigeonhole Principle, two of the Queens will be on squares of the same diagonal, and hence attack each other. This is a contradiction. Hence no quadrants may be empty. (Andy Liu)

2. Answer: yes, it is possible.

Suppose the stones are a, b, c, d grams in weight.

First solution.

For instance, the following 4 weighings suit:

1) one scale contains the stones a, b, and the other one contains the stones c, d;

2) one scale contains a, c, and the other one contains b, d;

3) one scale contains a, d, and the other one contains b, c;

4) one scale is empty, and the other one contains a, b, c, d.

Let $b = a + x$, $c = a + y$, $d = a + z$. First three weighings determine (by adding up their results) the difference between $3a + (b + c + d)$ and $2(b + c + d)$, that is, $x + y + z$, accurate to 1. The last weighing determines $a + b + c + d = 4a + (x + y + z)$ accurate to 1. Note that only one mistake is possible. Hence we know the difference between $(x + y + z)$ and $4a + (x + y + z)$, that is, $4a$, up to 1. Then we immediately determine a using division by 4. Similarly we find b, c, d.

Second solution.

For instance, the following 4 weighings suit:

1) one scale contains a, b, c, and the other one contains d;

2) one scale contains a, b, d, and the other one contains c;

3) one scale contains a, c, d, and the other one contains b;

4) one scale contains b, c, d, and the other one contains a.

Reading the balance, we obtain the following numbers (not more than one of them may be wrong):

$x = a + b + c - d$, $y = a + b - c + d$, $z = a - b + c + d$, $t = -a + b + c + d$.

Now we solve this system: for example, to find a, we add up three first equations and subtract the fourth one to get $4a = (x + y + z - t)$, whence $a = (x + y + z - t)/4$. Similarly we find b, c and d. Possibly we know just one of numbers x, y, z, t with an error of 1 gram. Hence in the expressions for a, b, c, d the numerators may be not divisible by 4. But their true values are recovered immediately (adding or subtracting 1 to get a numerator divisible by 4). This makes it possible to determine the weights of the stones.

3. Answer: 1/2.

Let us slightly paraphrase the problem. Extend median AD by its length beyond point D to obtain point D'. Then $CABD'$ is a parallelogram (because the diagonals of this quadrilateral are bisected by the intersection point), hence angle DAB equals angle $DD'C$. Since the sum of angles CAB and ACD' of the parallelogram equals 180°, angle CAB is obtuse iff angle ACD' is acute. The resulting statement: knowing only the length of side AD' (equal to the doubled length of AD), Elias managed to prove that in triangle CAD', the angles adjacent to side CD' are acute. For what ratio of sides AC and AD' is it possible?

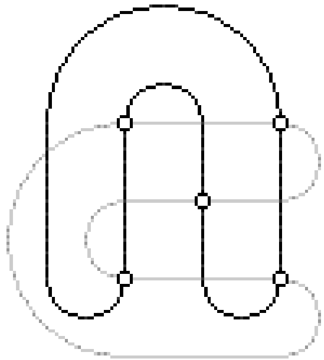
Clearly, if triangle CAD' is isosceles ($AC = AD' = 2AD$), then the angles adjacent to the base are acute (since they are equal and their sum is less than 180°). Thus the value $AD/AC = 1/2$ fits (and Elias's assertion is proven for this case).

If triangle CAD' is not isosceles ($AC \neq AD'$), then take for example the greater of sides AC and AD' as the hypotenuse, and the smaller one as a leg and construct triangle CAD' such that one of angles ACD' or $AD'C$ is right. Now complete parallelogram $CABD'$ and find triangle

CAB, which could well be drawn by Serge. In this triangle, either angle DAB is right or angle CAB is right. Thus Elias cannot prove his assertion for $AD/AC \neq 1/2$.

4. Answer: yes, it can.

See an example in the figure:



5. *First solution.* Expand the expression $(1+a_1)(1+a_2)\dots(1+a_n)$ to obtain the sum $1+(a_1+\dots+a_n)+(a_1a_2+\dots+a_{n-1}a_n)+(a_1a_2a_3+\dots a_{n-2}a_{n-1}a_n)+\dots+a_1a_2\dots a_n$. The first parentheses contain just the sum of $a_1, \dots, a_n$. The summands in the second parentheses are products of pairs of $a_1, \dots, a_n$. The summands in the third parentheses are products of triples of $a_1, \dots, a_n$, and so on. Clearly the sum in the second parentheses does not exceed $(a_1+\dots+a_n)^2$, the sum in the third parentheses does not exceed $(a_1+\dots+a_n)^3$, and so on. Hence our product does not exceed $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^n}=2-\frac{1}{2^n}<2$.

Second solution. Let us prove by induction that for all k from 1 to n we have $(1+a_1)(1+a_2)\dots(1+a_k)<1+2(a_1+\dots+a_k)$.

For $k=1$ the assertion is obvious ($1+a_1<1+2a_1$ since a_1 is positive).

The induction step is as follows.

Suppose that for some k, $k<n$, we have $(1+a_1)(1+a_2)\dots(1+a_k)<1+2(a_1+\dots+a_k)$. Multiply this inequality by $(1+a_{k+1})$ to get

$$(1+a_1)(1+a_2)\dots(1+a_{k+1})<(1+2(a_1+\dots+a_k))(1+a_{k+1})\leqslant$$
$$\leqslant 1+2(a_1+\dots+a_k)+a_{k+1}(1+2\cdot\frac{1}{2})=1+2(a_1+\dots+a_{k+1}).$$

The assertion is proved.

Take $k=n$ to obtain $(1+a_1)(1+a_2)\dots(1+a_n)<1+2(a_1+\dots+a_n)<1+2\cdot\frac{1}{2}=2$ as required.

Remark for advanced readers. In fact, we have a sharper inequality. It is possible to prove that for $a_1+\dots+a_n$ being constant, the expression $(1+a_1)(1+a_2)\dots(1+a_n)$ attains its maximum when $a_1, \dots, a_n$ are equal. Hence

$$(1+a_1)(1+a_2)\dots(1+a_n)\leqslant\left(1+\frac{1}{2n}\right)^n=\sqrt{\left(1+\frac{1}{2n}\right)^{2n}}.$$

As is known from the course of calculus, the radicand does not exceed the Neper constant (the Euler constant) $e=2,71828..$, whence $(1+a_1)(1+a_2)\dots(1+a_n)<\sqrt{e}=1,64...$

By the way, the inequality $\left(1+\frac{1}{2n}\right)^n\leq 2$ is equivalent to $\left(1-\frac{1}{2n+1}\right)^n\geq\frac{1}{2}$, which is an immediate consequence of the Bernoulli inequality.

6. Denote the meet of CC_1 and $A'B'$ by C', the midpoint of AC by M_B, and the midpoint of BC by M_A. Note that quadrilateral $CB'M_BC'$ is inscribed since $\angle CC'B'=\angle CM_BB'$. Similarly $CA'M_AC'$ is inscribed. Hence $\angle CC'M_B=\angle CC'M_A=90^\circ+\varphi$. Draw lines through vertices A and B, parallel to M_BC', M_AC' respectively. Suppose they meet at point C''. Using similarity, we see that C'' lies on line CC'. At the same time, CC' is a bisector in triangle $AC''B$. By the well-known fact that a bisector contains the midpoint of the corresponding arc of the circumcircle, we have $\angle AC_1B = 180^\circ - 2(90^\circ - \varphi) = 2\varphi$.

7. a) Let b_n be 1 if the greatest odd divisor of n has residue 1 modulo 4, and let it be -1 otherwise. Then $a_n = b_1 + b_2 + \cdots + b_n$.

Let us write the sequence of numbers b_n in rows so that the length of the first row be 1, the length of the second row be 2, ..., the length of the kth row be 2^{k-1}:

$b_1,$
$b_2, b_3,$
$b_4, b_5, b_6, b_7,$
$\ldots,$
$b_{2^{k-1}}, \ldots, b_{2^k-1},$
$\ldots$

The first five rows are as follows:

$1,$
$\mathbf{1}, -1,$
$\mathbf{1}, 1, -\mathbf{1}, -1,$
$\mathbf{1}, 1, \mathbf{1}, -1, -\mathbf{1}, 1, -\mathbf{1}, -1,$
$\mathbf{1}, 1, \mathbf{1}, -1, \mathbf{1}, 1, -\mathbf{1}, -1, -\mathbf{1}, 1, \mathbf{1}, -1, -\mathbf{1}, 1, -\mathbf{1}, -1,$

Note that each row beginning with the third one is obtained from the preceding row by inserting alternating 1 and -1 between its elements. (For clearness, numbers from the preceding row are printed in bold face.) Indeed, $b_{2m} = b_m$ for any m; the $(k+1)$th row has the form

$b_{2^k}, b_{2^k+1}, b_{2^k+2}, b_{2^k+3}, b_{2^k+4}, b_{2^k+5}, \ldots$

which reduces to

$b_{2^{k-1}}, 1, b_{2^{k-1}+1}, -1, b_{2^{k-1}+2}, 1, \ldots$

We have used the fact that 2^k is a multiple of 4 for $k > 1$.

Let us prove that the total sum of the kth row is 0 for $k > 1$. We shall use induction in k. The base $b_2 + b_3 = 0$ is obvious.

Assume that the sum of the numbers in the kth row is 0. Since the elements of the $(k+1)$th row are elements of the kth row plus an even number of alternating 1 and -1's, the total sum of the $(k+1)$th row is 0.

This easily implies (again by induction) that $a_{2^k-1} = 1$ for any $k > 0$, so the number 1 occurs in the sequence (a_n) infinitely many times.

b) Now we will prove that, for $k > 1$, the kth row has some initial part with sum of elements equal to $k-1$. We will use induction. The base is again obvious.

Denote by m_k the length of the initial part of the kth row with sum $k-1$. For even m_k, take $m_{k+1} = 2m_k - 1$, and for odd m_k, take $m_{k+1} = 2m_k$. Consider first m_{k+1} elements of the $(k+1)$th row. In any case, clearly, the initial part of the $(k+1)$th row of length m_{k+1} is obtained by inserting an odd number of alternating 1 and -1's between the elements of the initial part of the kth row. Thus the sum increases by 1 as required.

By the proof of part a), we have $a_{2^{k-1}-1} = 1$. This corresponds to the end of the $(k-1)$th row. Hence $a_{2^{k-1}-1+m_k} = k$. Since a_n changes by 1 (in some direction) as n increases by 1, we conclude that a_n necessarily takes all integer values from 1 to k as n takes integer values from $2^{k-1} - 1$ to $2^{k-1} - 1 + m_k$. This implies that each positive integer occurs in the sequence (a_n) infinitely many times.

Criteria for examination

As usual, "+" marks any correct solution, "+−" marks a solution with a significant but easily removable gap, "−+" marks a wrong solution however containing a significant advance, "−" marks a wrong solution, and "0" means that the problem has not been written down. The marks "+." "−." mark less significant gaps (advances) than "±" and "∓". The mark "+/2" is used in specific cases when the text contains a valid idea however insufficiently developed to consider the problem as solved. This mark is used also in the case when the problem naturally divides into two parts and just one of them has been solved. If the judges wish to emphasize an unusual progress of the student (conciseness of the solution, its beauty, an extension of the result of the problem etc.), this is marked by "+!".

In examination of a large number of papers, there arise typical cases which need a specification whether a gap (an advance) is significant. These cases are described below.

O-level, Juniors

Problem 1.

$\mp$ The boxes are ordered by increasing number of pencils in them, and pencils are chosen consecutively. But the key assertion is lacking: that each box contains more pencils than it was chosen from the preceding boxes. For example, it is only said that the box at hand contains pencils of another color than the preceding box.

Problem 2.

$\pm$ An error in calculation, resulting in a wrong answer, but such that its correction leads to the correct answer.

$\mp$ *and better* The idea of considering pairs of numbers with difference 50 and taking a single number from each pair.

$-$. Only correct answer 2525 or the answer and specific cases of the choice of numbers.

Problem 3.

$-$ A solution only for the case of a regular triangle.

Problem 5.

$-$ All meetings only at the start point.

$\mp$ The idea that meetings will repeat after the same interval of time.

A-level, Juniors

Problem 1.

$\pm$ The problem is solved but there is no proof for the assertion that if a corner square does not contain queens then each square adjacent (by side) to it contains 50 queens.

$\mp$ Only the proof for the assertion that if a corner square does not contain queens then each square adjacent (by side) to it contains 50 queens.

Problem 2.

$\pm$ Only a correct example of weighings without a proof.

Problem 3.

$\pm$ The assertion that if the ratio exceeds 1/2 then some angle can be acute, if the ratio is less than 1/2 then another angle can be obtuse, and the ratio 1/2 fits. In all three cases there are attempts to use moving of the third vertex of the triangle but the argument is not strict.

$+/2$ Only the correct ratio is determined (with a proof), or there is only a proof of Elias's assertion for this ratio.

$\mp$ The inequality is proved in a single direction (for instance, that AD/AC does not exceed 1/2).

$\mp$ A half of Elias's assertion is proved (for instance, that $\angle BAC$ is acute) for the ratio 1/2.

$-$. Only the answer.

Problem 4.

$-$. It is proved that the roads of the same color form a cycle without self-intersections.

$-$. It is proved that each road must intersect a road of another color.

Problem 5.

The mark is not decreased for the lack of the proof that the sum of a geometrical progression with ratio 1/2 and the first term 1/2 is less than 1.

$+$ The mark is not decreased for referring to Cauchy inequality or to the fact that the product attains its maximum in the case of equal factors.

± The assertion without a correct proof that the problem reduces to the case $a_1 = \cdots = a_n$, and a solution for this case.

∓ The problem is reduced to the case $a_1 = \cdots = a_n$ without further advance.

∓ The idea that after removing parenthesis, the terms can be grouped and the sum can be estimated by a geometrical progression, but this is not proved.

− Only the case $n = 2$ is considered.

Problem 6.

−. Only the answer.

Problem 7a).

−. The assertion without a proof that if $a_n = 1$ then $a_{2^n+1} = 1$ as well.

−. The assertion without a proof that the members of the sequence numbered by $2^n - 1$ are equal to 1.

∓ The assertion that for odd numbers, the residues modulo 4 of the greatest odd divisor alternate and that for an even number and its half, the residues modulo 4 of the greatest odd divisor coincide, with no further advance.

± The same as in the last item plus the assertion: hence the sum of residues modulo 4 of greatest odd divisors of odd numbers equals 0, the similar argument for even numbers and the assertion (without an argument) that 1 will occur infinitely many times.

Problem 7b).

∓ For some segment $[a_k, b_k]$ obtained by doubling the preceding segment, there is an attempt to prove that it contains the number k, but this is wrong for this segment.

Solutions for the problems of the Autumn round of the 30th Tournament of Towns

A-level, Seniors

1. Answer: 5/4.

Consider 2nd, 4th, and 6th vertical lines and similarly 2nd, 4th, and 6th horizontal lines. These lines divide the board into 16 rectangles, each being divided into 4 cells colored as on a chessboard. If in each of these rectangles, the ratio of the total area of white cells to that of black cells does not exceed 5/4, then this is true for the whole board (indeed, if in the ith rectangle, a_i is the area of white cells, b_i is the area of black cells, and $a_i \leq (5/4)b_i$, then add up all these inequalities to obtain the required inequality for the whole board).

Now consider a single rectangle divided into 4 cells. Let the lengths of its sides be x (horizontal) and y (vertical). Clearly the length of the horizontal side of some white cell at least $x/2$. We may assume that it is the upper left cell. We can move down the horizontal line dividing the rectangle until the ratio of areas of the upper left (white) cell and the lower left (black) cell becomes equal to 2. The condition that the area of any white cell does not exceed the doubled area of any black cell still holds, and the total area of white cells could only increase (or not change). Further shifting of the line would violate this condition. Now we may similarly move to the right the vertical line dividing the rectangle (until the area of the upper left cell becomes equal to the doubled area of the upper right cell). Now the sides of the upper left cell are equal to $2x/3$ and $2y/3$, the total area of white cells is $4xy/9 + xy/9 = 5xy/9$, and that of black cells is $xy - 5xy/9 = 4xy/9$. Now the ratio of these areas is 5/4, and the initial ratio does not exceed it.

It remains to give an example when the required ratio equals 5/4 in each of 16 rectangles. Divide the board so that these 16 rectangles are equal squares, the upper left cell of each square is a square, and its sides are twice as long as the sides of the lower right cell. Clearly this dissection is possible and gives the required example.

2. Answer: not necessarily true.

Below is some example which is described rather easily. Suppose in the space three directions are fixed: up and down, to the right and to the left, back and forth. Fill in the space by cubes in a usual way. Now choose some cube. Six infinite columns, one cube thick, are adjacent to it: two parallel columns at the left and at the right directed up and down, two columns from the front and from behind directed to the right and to the left, and two columns from below and from above directed back and forth. Shift each columns in its direction by the half-side of the cube (shift first two layers up, the next two to the left, and the next two forth). The space is still filled up but the chosen cube has no entire face in common with any other cube.

3. The second player can always win. Consider two cases.

Suppose $N > 2$ is odd.

We will prove that the second player can join two greatest piles at each move.

At the first move, the first player necessarily makes a pile of two nuts, and after that the second player makes a pile of three nuts. Then the situation is as follows: the greatest pile contains an odd number of nuts, and each of other piles contains a single nut.

Now the first player has two possibilities. Either he makes a pile of two nuts, and then the second player adds it to the greatest pile. Or he increases the greatest pile by one nut, and then it contains an even number of nuts. Since N is odd, there remains at least one pile consisting of a single nut, and the second player can add this nut to the greatest pile, so that we have the described situation again.

We see that the second player always can make a move. Finally the piles will be over, and the first player will lose.

Now suppose $N > 2$ is even.

Note that each move of each player decreases the number of piles by 1. Therefore, the number of piles after each move of the first player is odd. Let the second player act as in the case of odd N until there are only three piles before his move.

If each of two "little" piles contains a single nut, he just unites the "little" piles and leaves two piles with even number of nuts, and the first player loses. If the "little" piles contain one and two nuts then he adds one nut to the greatest pile (and the first player loses as well).

4. Let O be the point of meet for the diagonals of trapezoid $ABCD$.

Since quadrilateral A_1BCD is inscribed, we have $BO \cdot OD = A_1O \cdot OC$ (by similarity of triangles BOA_1 and COD). Similarly, $BO \cdot OD = AO \cdot OC_1$, $AO \cdot OC = BO \cdot OD_1$, and $AO \cdot OC = B_1O \cdot OD$.

The first two equations imply $OC/AO = OC_1/A_1O$, the two others imply $BO/OD = B_1O/OD_1$.

Note that parallelism of sides BC and AD in trapezoid $ABCD$ may be expressed in the form $BO/OD = OC/AO$ (by similarity of triangles BOC and DOA).

But then (by the above) $BO_1/OD_1 = OC_1/A_1O$, so sides B_1C_1 and A_1D_1 are parallel.

Similarly we verify that non-parallelism of sides AB and CD implies non-parallelism of sides A_1B_1 and C_1D_1.

Thus $A_1B_1C_1D_1$ is a trapezoid as required.

5. Let b_n be 1 if the greatest odd divisor of n has residue 1 modulo 4, and -1 otherwise. Then $a_n = b_1 + b_2 + \dots + b_n$.

We will prove by induction that $a_{2^k-1} = 1$. The base is correct. Suppose $a_{2^k-1} = 1$. Note that $b_m = b_{2^k+m}$ for $m < 2^k$ and $m \neq 2^{k-1}$. Indeed, if $m = a \cdot 2^l$, where a is odd, then $2^k + m = (2^{k-l} + a) \cdot 2^l$, where $l < k$, so $2^{k-l} + a$ is the greatest odd divisor of $2^{k-1} + m$. Since $m \neq 2^{k-1}$, we have $k > l + 1$, so 2^{k-l} is a multiple of 4.

For $m = 2^{k-1}$, we have $m + 2^k = 3 \cdot 2^{k-1}$, hence $b_m = 1$ and $b_{m+2^k} = -1$. Therefore, $(b_{2^k+1} + b_{2^k+2} + \dots + b_{2^{k+1}-1}) = (b_1 + b_2 + \dots + b_{2^k-1}) - 2$.

Then, by the induction hypothesis and by the above,

$$b_1 + b_2 + \dots + b_{2^{k+1}-1} = (b_1 + b_2 + \dots + b_{2^k-1}) + b_{2^k} + (b_{2^k+1} + b_{2^k+2} + \dots + b_{2^{k+1}-1}) = \\ = 2 \cdot (b_1 + b_2 + \dots + b_{2^k-1}) - 2 + b_{2^k} = 2 \cdot 1 - 2 + 1 = 1.$$

Now we will prove by induction that in the segment $1 \dots 2^k - 1$ the sequence a_n takes value k. For the first two segments this is true. Suppose $a_x = n - 1$ where $x < 2^{n-1}$. We will prove that $a_{2^n+x} = n + 1$. Indeed, by induction and by the above

$$b_1 + b_2 + \dots + b_{2^n+x} = (b_1 + b_2 + \dots + b_{2^n-1}) + b_{2^n} + (b_{2^n+1} + b_{2^n+2} + \dots + b_{2^n+x}) = \\ = 1 + 1 + (b_1 + b_2 + \dots + b_x) = n + 1.$$

Note that in the segment $2^n - 1 \dots 2^n + x$ the sequence takes all values from 1 to $n+1$. Hence each number occurs infinitely many times in the sequence.

Remark: the solution implies that $a_{2^n+2^{n-2}+2^{n-4}+\dots} = n + 1$.

6. We may assume that the leading coefficient of $P(x)$ equals 1. If the degree of $P(x)$ is even, then we have $P(x) > 0$ for sufficiently great absolute values of x, and there exist only finitely many pairs of integers m, n such that $P(m) + P(n) = 0$. Hence the degree of $P(x)$ is odd (we will use this without reminding).

Note that $P(x)$ tends to infinity as $x \to \infty$, and monotonously increases starting from some x. Similarly, $P(x)$ tends to $-\infty$ as $x \to -\infty$, and monotonously decreases starting from some x.

At the same time, for a given n, there exist at most finitely many values of m such that $P(m) = -P(n)$ (because a polynomial can take the same specific value only in a finite number of points, not exceeding its degree).

The above implies that, for any given number C, there exist pairs of numbers m, n such that $P(m) + P(n) = 0$, m and n have different sign, and their absolute values exceed C.

Suppose the degree of $P(x)$ is k, and $P(x) = x^k + ax^{k-1} + \dots$ (here and in the sequel, dots denote the terms of lower degrees). Clearly there exists d such that $P(x - d)$ has the form $x^k + bx^{k-2} + \dots$, that is, the coefficient at the $(k-1)$th degree is zero. Indeed, $P(x - d) =$

$(x-d)^k + a(x-d)^{k-1} + \dots = x^k - kdx^{k-1} + ax^{k-1} + \dots$, and we can take a/k for d. We will prove that point $(d;0)$ is the center of symmetry for the graph. Denote $P(x-d)$ by $Q(x)$. It suffices to prove that $Q(x) = -Q(-x)$ for all x.

Thus $Q(x) = x^k + bx^{k-2} + \dots$ and we know that the equation $Q(m) + Q(n) = 0$ has infinitely many solutions m, n such that $m-d$ and $n-d$ are integers. Take one of these solutions $m > 0$, $n < 0$ with a sufficiently large absolute value. We will prove that $|m| = |n|$. Suppose $|m| < |n|$, for instance, $n = -m-1$. Then

$$Q(m) + Q(n) = Q(m) + Q(-m-1) = \\ = m^k + bm^{k-2} + \dots + (-m-1)^k + b(-m-1)^{k-2} + \dots = \\ = -km^{k-1} + R(m),$$

where $R(x)$ is some polynomial of degree $k-2$. If m is sufficiently large then km^{k-1} is much greater than $|R(m)|$, so the sum $-km^{k-1} + R(m)$ is negative. When $|n|$ increases, the sum $Q(m) + Q(n)$ decreases (because $Q(n)$ decreases). Hence $|m| < |n|$ is impossible for sufficiently large absolute values of m and n. Similarly $|m| > |n|$ is impossible. Hence there exist infinitely many numbers m such that $Q(m) + Q(-m) = 0$, that is, the polynomial $Q(x) + Q(-x)$ has infinitely many roots. This is possible only if it is zero polynomial, that is, we have the identity $Q(x) + Q(-x) = 0$. But this just means that the graph of $Q(x)$ is symmetrical in point $(0;0)$, and so the graph of $P(x)$ is symmetrical in $(d;0)$.

7. a) Answer: yes, he can.

In the first 29 attempts let Victor answer as follows: in the kth attempt he answers " yes" to the kth question and "no" to all other questions. Note that any two attempts among the first 29 ones differ just in two answers. Hence the numbers of correct answers in these two attempts either are equal or differ by 2. If in some two attempts, say ith and jth, the numbers of correct answers differ by 2, then we immediately know correct answers to the ith and the jth question. Then we may compare, for instance, the ith attempt with all the others to know correct answers to questions 1–29. Now we can learn the answer to question 30 from the first attempt: it suffices to count the number of correct answers to the first 29 questions and to compare it with the information on the number of correct answers in the first attempt.

In the remaining case, the numbers of correct answers in any two attempts from the first to the 29th one are equal. This means that the answers to questions 1–29 are identical. But then the information on the number of correct answers in the first attempt shows whether these are "yes" or "no" (in the first case the number of correct answers is 1 or 2, in the second case it is 28 or 29). After that we learn the answer to question 30 (also from the first attempt).

Note that this method works if the total number of questions in the test is at least 5.

b) Divide the questions of the test into groups of 5 questions each (questions 1–5, 6–10 etc.). In the first attempt we answer "no" to all questions.

Now we will show how to learn correct answers to all questions of the first group during 4 next attempts. We answer "no" to questions from 6 to 30, and we answer to questions from 1 to 5 as follows:

2nd attempt: "yes" to all five questions;
3rd attempt: "no" to questions 1 and 2 ("yes" to the remaining three questions);
4th attempt: "no" to questions 1 and 3 ("yes" to the remaining three questions);
5th attempt: "no" to questions 1 and 4 ("yes" to the remaining three questions).

Note that the information on the number of correct answers in the first and in the second attempt shows, for instance, the number of correct answers to questions from 1 to 5 in the first attempt.

If in the second attempt the answers to questions 1 and 2 are both correct or both incorrect then after the third attempt we know answers to questions 1 and 2, after the fourth attempt to question 3, after the fifth attempt to question 4, and then we know the answer to question 5 as well (since we know the number of correct answers to questions 1 – 5 in the second attempt). Similarly for questions 1, 3 and 1, 4. It remains to consider the case when in the second attempt just one of the answers to questions 1, 2 is correct, and similarly for answers to questions 1, 3 and to questions 1, 4. That is, in the second attempt the answers to questions 1, 2, 3 were either all correct or all incorrect.

But we already know what answers to questions 1 – 5 are more numerous, correct or incorrect (if we answer "no" to all of them). So we know answers to questions 2, 3, 4, hence to question 1, and then to question 5.

Similarly we learn answers to questions of the 2nd, 3rd, 4th and 5th group (spending 4 attempts each time). Thus $1 + 4 \cdot 5 = 21$ attempts enable us to learn answers to questions 1 — 25. Now observe that we can learn the answers to the questions of the last group in 3 attempts since we already know the number of correct answers to these questions in the first attempt. Thus 24 attempts suffice.

Criteria for examination.

As usual, "+" marks any correct solution, "+−" marks a solution with a significant but easily removable gap, "−+" marks a wrong solution however containing a significant advance, "−" marks a wrong solution, and "0" means that the problem has not been written down. The marks "+." "−." mark less significant gaps (advances) than "±" and "∓". The mark "+/2" is used in specific cases when the text contains a valid idea however insufficiently developed to consider the problem as solved. This mark is used also in the case when the problem naturally divides into two parts and just one of them has been solved. If the judges wish to emphasize an unusual progress of the student (conciseness of the solution, its beaty, an extension of the result of the problem etc.), this is marked by "+!".

In examination of a large number of papers, there arise typical cases which need a specification whether a gap (an advance) is significant. These cases are described below.

O-level, Seniors.

Problem 2.

+− The values $x_1 = 1$ and $x_2 = 0$ have been found and the equality $x_3 + \cdots + x_n = 0$ has been proved but the answer is lacking (the solution is incomplete)

−+ The proof for $x_3 + \cdots + x_n = 0$ with no further advance

−+ The values $x_1 = 1$ and $x_2 = 0$ have been found with no further advance (no idea that $x_3 + \cdots + x_n = 0$)

Problem 3.

The consideration of the single case of regular 30-gon is evaluated not better than −+

+. The presented argument works only for the case when the center of the circle lies inside 30-gon

−. The idea is declared but not realized to make the length of A_1A_2 and the area of OA_1BA_2 numerically equal

Problem 5.

− For coloring of specific pictures only or for coloring of small parts of adjacent rectangles (local construction)

A-level. Seniors

Problem 1.

+. A correct argument for the case of the rectangle 2×2, an example for the board 8×8, and the assertion that if the ratio does not exceed 5/4 in each of 16 rectangles 2×2 then the same is true for the large rectangle (the proof is not presented as obvious)

+− A correct argument for the case of the rectangle 2×2, an example for the board 8×8, but a wrong proof for the assertion that if the ratio does not exceed 5/4 in each of 16 rectangles 2×2, then the same is true for the large rectangle.

+− The estimate without an example.

−+ A correct argument only for the rectangle 2×2

Problem 2.

+− A correct example without sufficient explanation (for example, no explanation how to extend the construction to the whole space)

Problem 3.

+− A correct argument only for the even case

−+ A correct argument only for the odd case

Problem 5.

−+ The proof for $a_{2^n-1} = 1$

Problem 7a).

+− A correct algorithm with no explanation

+− Minor gaps in the algorithm (for instance, an easily removable error in consideration of some subcase)

Problem 7b).

+− Minor gaps in the algorithm (in this case, 7a)b) was evaluated as + +−)