INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

JUNIOR PAPER: YEARS 8, 9, 10

Tournament 31, Northern Autumn 2009 (A Level): Solutions

1. **Alternative 1**

   Pour from each jar exactly one tenth of what it initially contains into each of the other nine jars. At the end of these ten operations, each jar will contain one tenth of what is inside each jar initially. Since the total amount of milk remains unchanged, each jar will contain one tenth of the total amount of milk. (Han Hyung Lee)

   Alternative 2

   Let k be the number of jars which contain the smallest amount of milk. If $k = 10$, there is nothing to do. Suppose $k < 10$. Let each of these k jars contain m units of milk. There is another jar which contains the next smallest amount of milk, say n units.

   Pour $\frac{n-m}{10}$ units from the jar containing n units into each of the other nine jars. Then each of the k jars with m units before now has

   $$m + \frac{n-m}{10} = \frac{9m+n}{10}$$

   units of milk.

   The jar with n units before now has

   $$n - \frac{9(n-m)}{10} = \frac{9m+n}{10}$$

   units also. In either case, the value of k has been increased by 1. Performing this procedure at most 9 times, we can raise k to 10. (A Liu)

2. Assign spatial coordinates to the unit cubes, each dimension ranging from 1 to 10. If all cubes are in the same colour orientation, there is nothing to prove. Hence we may assume that (i, j, k) and $(i + 1, j, k)$ do not. Since they share a left-right face, let the common colour be red. We may assign blue to the front-back faces of (i, j, k). Then its top-bottom faces are white, the front-back faces of $(i+1, j, k)$ are white and the top-bottom faces of $(i + 1, j, k)$ is blue.

   Now $(i, j + 1, k)$ share a white face with (i, j, k) while $(i + 1, j + 1, k)$ share a blue face with $(i + 1, j, k)$. Since $(i, j + 1, k)$ and $(i + 1, j + 1, k)$ share a left-right face, the only available colour is red. It follows that the $1\times2\times10$ block with $(i, 1, k)$ and $(i+1, 1, k)$ at one end and $(i, 10, k)$ and $(i + 1, 10, k)$ at the other has 1×10 faces left and right which are all red.

   Similarly, if we carry out the expansion vertically, we obtain a $2 \times 10 \times 10$ black with 10×10 faces left and right which are all red. Finally, if we carry out the expansion sideways, we will have the left and right faces of the large cube all red. (A Liu)

3. Suppose $a = b$. Then $a + a^2 = a(a+1)$ is a power of 2, so that each of a and $a+1$ is a power of 2. This is only possible if $a = 1$. Suppose $a \neq b$. By symmetry, we may assume that $a > b$, so that $a^2 + b > a + b^2$. Since their product is a power of 2, each is a power of 2.

Let $a^2 + b = 2^r$ and $a + b^2 = 2^s$ with $r > s$. Then

$$2^s(2^{r-s} - 1) = 2^r - 2^s = a^2 + b - a - b^2 = (a-b)(a+b-1).$$

Now $a - b$ and $a + b - 1$ have opposite parity. Hence one of them is equal to 2^s and the other to $2^{r-s} - 1$. If

$$a - b = 2^s = a + b^2,$$

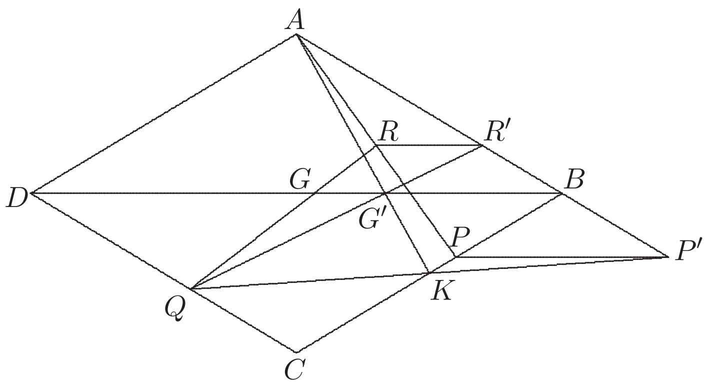
then $-b = b^2$. If

$$a + b - 1 = 2^s = a + b^2,$$

then $b - 1 = b^2$.

Both are contradictions. Hence there is a unique solution $a = b = 1$. (A Liu)

4. Extend AB to P' so that $BP' = BP = CQ$. Then $BP'CQ$ is a parallelogram so that $P'Q$ and BC bisect each other at a point K.



Let AK intersect BD at G' and let QG' intersect AB at R'. Since K is the midpoint of BC, its distance from BD is half the distance of C from BD, which is equal to the distance of A from BD. It follows that $AG' = 2KG'$.

Since K is the midpoint of $P'Q$, G' is the centroid of triangle $AP'Q$. Hence $QG' = 2R'G'$ and R' is the midpoint of AP'. Let R be the midpoint of AP and let QR intersect BD at G. Then RR' is parallel to PP', which is in turn parallel to BD.

Hence $QG = 2RG$ so that G is the centroid of triangle APQ. (A Liu)

5. (a) Suppose $n + 1 = k^2$ for some positive integer k. We take the lightest k objects with total weight

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

grams. The average weight of the remaining objects is

$$\frac{(k+1) + (k^2 - 1)}{2} = \frac{k(k+1)}{2}$$

grams also.

(b) The total weight of the n objects is

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

grams. Let T grams be the total weight of the k chosen objects. This is also the average weight of the remaining $n-k$ objects. Hence

$$\frac{n(n+1)}{2} = T(n-k+1).$$

Now

$$2T(n-k+1) = n(n+1) > n^2 + n - k^2 + k = (n+k)(n-k+1),$$

so that $2T > n+k$. If we choose the lightest k objects, then T attains its maximum value $\dfrac{(k+1)+n}{2}$, so that $2T \leq n+k+1$.

It follows that we must have $2T = n+k+1$, and we must take the lightest k objects. Then

$$\frac{n+k+1}{2} = T = 1 + 2 + \cdots + k = \frac{k^2+k}{2},$$

so that $n+1 = k^2$. (Central Jury)

6. Partition the infinite chessboard into $n \times n$ subboards by horizontal and vertical lines n units apart. Within each subboard, assign the coordinates (i, j) to the square at the i-th row and the j-th column, where $1 \leq i, j \leq n$.

Whenever an $n \times n$ cardboard is placed on the infinite chessboard, it covers n^2 squares all with different coordinates. The total number of times squares with coordinates (1,1) is covered is 2009. Since 2009 is odd, at least one of the squares with coordinates (1,1) is covered by an odd number of cardboards. The same goes for the other $n^2 - 1$ coordinates.

Hence the total number of squares which are covered an odd number of times is at least n^2. (Olga Ivrii)

7. We construct a graph, with the vertices representing the islands and the edges representing connecting routes. The graph may have one or more connected components. Since the total number of vertices is odd, there must be a connected component with an odd number of vertices.

Olga chooses from this component the largest set of independent edges, that is, edges no two of which have a common endpoint. She will colour these edges red. Since the number of vertices is odd, there is at least one vertex which is not incident with a red vertex. Olga will start the tour there.

Suppose Max has a move. It must take the tour to a vertex incident with a red edge. Otherwise, Olga could have colour one more edge red. Olga simply continues the tour by following that red edge. If Max continues to go to vertices incident with red edges, Olga will always have a ready response.

Suppose somehow Max manages to get to a vertex not incident with a red edge. Consider the tour so far. Both the starting and the finishing vertices are not incident with red edges. In between, the edges are alternately red and uncoloured. If Olga interchanges the red and uncoloured edges on this tour, she could have obtained a larger independent set of edges.

This contradiction shows that Max could never get to a vertex not incident with red edges, so that Olga always wins if she follows the above strategy. (Central Jury)

INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11, 12

Tournament 31, Northern Autumn 2009 (A Level): Solutions

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. **Alternative 1**

A pirate who owes money is put in group A, and the others are put in group B. Each pirate in group A puts the full amount of money he owes into a pot, and the pot is shared equally among all 100 pirates. For each pirate in group B, each of the 100 pirates puts $\frac{1}{100}$-th of the amount owed to him in a pot, and this pirate takes the pot. We claim that all debts are then settled.

Let a be the total amount of money the pirates in group A owe, and let b be the total amount of money owed by the pirates in group B. Clearly, $a = b$. Each pirate in group A pays off his debt, takes back $\frac{a}{100}$ and then pays out another $\frac{b}{100}$. Hence he has paid off his debt exactly. Each pirate in group B takes in $\frac{a}{100}$, pays out $\frac{b}{100}$ and then takes in what is owed him. Hence the debts to him have been settled too. (Wen-Hsien Sun)

Alternative 2

Let M units be the maximum amount of money won by one pirate. Such a pirate brings no money to the changing room. A pirate who wins N units where $N < M$ brings to the changing room an amount of money equal to $M - N$. A pirate who loses N units brings to the changing room an amount of money equal to $M + N$. Let k be the number of pirates who has the smallest amount of money in the changing room. If $k = 100$, there is nothing to do.

Suppose $k < 100$. Let each of these k pirates have m units of money. There is another pirate who has the next smallest amount of money, say n units.

He gives $\dfrac{n-m}{100}$ units to each of the other 99 pirates. Then each of the k pirates with m units before now has

$$m + \frac{n-m}{100} = \frac{99m+n}{10}$$

units of money. The pirate with n units before now has

$$n - \frac{99(n-m)}{100} = \frac{99m+n}{100}$$

units also.

Thus the value of k has been increased by 1. Performing this procedure at most 99 times, we can raise k to 100. Each of the pirates now has M units of money, meaning that all debts have been collected or paid accordingly. (A Liu)

2. Let the given rectangle R have length m and width n with $m > n$. Contract the length of R by a factor of $\frac{n}{m}$, resulting in an $n \times n$ square. For each of the N rectangle in R, the corresponding rectangle in S has the same width but shorter length.

Thus we can cut the former into a primary piece congruent to the latter, plus a secondary piece. Using S as a model, the N primary pieces may be assembled into an $n \times n$ square while the N secondary pieces may be assembled into an $(m-n) \times n$ rectangle.

(Rosu Cristina and Jonathan Zung, independently)

3. Let the points of tangency to the sphere of AB, AC, DB and DC be k, L, M and N respectively.

The line KL intersects the line BC at some point P not between B and C. By the converse of the undirected version of Menelaus' Theorem,

$$1 = \frac{BP}{PC} \cdot \frac{CL}{LA} \cdot \frac{AK}{KB} = \frac{BP}{PC} \cdot \frac{CL}{KB}$$

since $LA = AK$. Since $CL = CN$, $KB = MB$ and $ND = DM$,

$$1 = \frac{BP}{PC} \cdot \frac{CN}{MB} = \frac{BP}{PC} \cdot \frac{CN}{ND} \cdot \frac{DM}{MB}.$$

By the undirected version of Menelaus' Theorem, P, M and N are collinear. It follows that K, L, M and N are coplanar, so that KN intersects LM. Similarly, the line joining the points of tangency to the sphere of AD and BC also intersects KN and LM. Since the three lines are not coplanar, they must intersect one another at a single point.

(A Liu)

4. Define $f(n) = 111\ldots1$ with n 1s and $f(0) = 1$ so that [0]!=1. Define

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

for $0 \le k \le n$.

We use induction on n to prove that $\begin{bmatrix} n \\ k \end{bmatrix}$ is always a positive integer for all $n \ge 1$. For $n = 0$,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{[0]!}{[0]![0]!} = 1.$$

Suppose the result holds for some $n \ge 0$. Consider the next case.

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k \end{bmatrix} &= \frac{[n+1]!}{[k]![n+1-k]!} \\
&= \frac{[n]!f(n+1)}{[k]![n+1-k]!} \\
&= \frac{[n]!f(n+1-k)10^k}{[k]![n-k]!f(n+1-k)} + \frac{[n]!f(k)}{[k-1]![n+1-k]!} \\
&= 10^k \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}.
\end{aligned}$$

Since both terms in the last line are positive integers, the induction argument is complete. In particular, for any positive integers m and n,

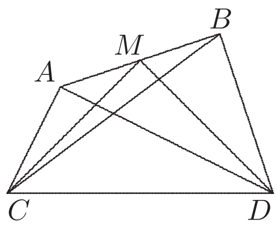
$$\left[\begin{array}{c} m+n \\ m \end{array}\right] = \frac{[m+n]!}{[m]![n]!}$$

is a positive integer, so that $[m+n]!$ is divisible by $[m]![n]!$. (Jonathan Zung)

5. Denote the area of a polygon P by $[P]$. We first establish a preliminary result.

Lemma.

Let M be the midpoint of a segment AB which does not intersect another segment CD.



Then

$$[CMD] = \frac{[CAD] + [CBD]}{2}.$$

Proof:

Since M is the midpoint of AB, we have

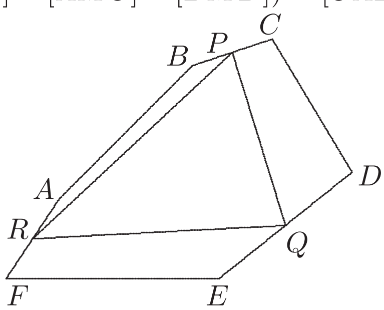
$$[CAD] = [ABDC] - [BAD] = [ABDC] - 2[BMD]$$

and

$$[CBD] = [ABDC] - [ABC] = [ABDC] - 2[AMC].$$

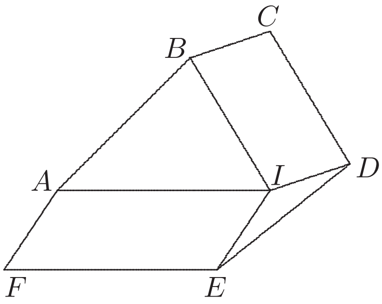
Hence

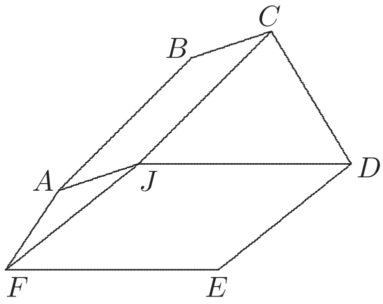
$$2[CMD] = 2([ABDC] - [AMC] - [BMD]) = [CAD] + [CBD].$$

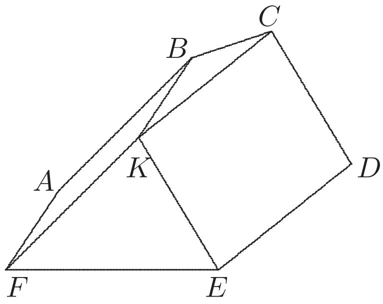


Returning to the problem, let P, Q and R be the respective midpoints of BC, DE and FA. By the Lemma, we have

$$\begin{aligned}
[PQR] &= \frac{1}{2}([BQR] + [CQR]) \\
&= \frac{1}{4}([BDR] + [BER] + [CDR] + [CER]) \\
&= \frac{1}{8}([BAD] + [BFD] + [BAE] + [BFE] + [CAD] + [CFD] + [CAE] + [CFE]).
\end{aligned}$$







Let I be the point such that ABI is congruent to XYZ. Then $BCDI$ and $EFAI$ are parallelograms. Since $ABCDEF$ is convex, I is inside the hexagon. Hence

$$[XYZ] < [ABCDEF].$$

Note that the distance of D from AB is equal to the sum of the distances from C and I to AB, Hence

$$[BAD] = [BAC] + [BAI] = [BAC] + [XYZ].$$

Similarly,

$$[BAE] = [BAF] + [XYZ].$$

Let J and K be the points such that JCD and FKE are congruent to XYZ. Then we have

$$\begin{aligned} [ACD] &= [BCD] + [XYZ], \\ [FCD] &= [ECD] + [XYZ], \\ [BFE] &= [AFE] + [XYZ] \\ \text{and} \quad [CFE] &= [DFE] + [XYZ]. \end{aligned}$$

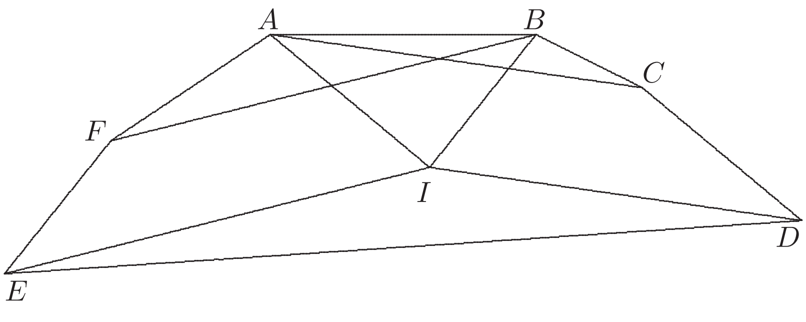
It follows that

$$[PQR] = \frac{1}{8}(2[ABCDEF] + 6[XYZ]) > [XYZ].$$

(Central Jury)

Remark:

The solution above makes a reasonable assumption that XYZ and $ABCDEF$ are in the same orientation. If they are not, the first of the three diagrams above may look like the one below, and minor modifications to the argument are necessary. However, this complication is a mere detraction to an already very nice problem.



6. We construct a graph, with the vertices representing the islands and the edges representing connecting routes. The graph may have one or more connected components. Since the total number of vertices is odd, there must be a connected component with an odd number of vertices.

Olga chooses from this component the largest set of independent edges, that is, edges no two of which have a common endpoint. She will colour these edges red. Since the number of vertices is odd, there is at least one vertex which is not incident with a red vertex. Olga will start the tour there.

Suppose Max has a move. It must take the tour to a vertex incident with a red edge. Otherwise, Olga could have colour one more edge red. Olga simply continue the tour by following that red edge. If Max continues to go to vertices incident with red edges, Olga will always have a ready response.

Suppose somehow Max manages to get to a vertex not incident with a red edge. Consider the tour so far. Both the starting and the finishing vertices are not incident with red edges. In between, the edges are alternately red and uncoloured. If Olga interchanges the red and uncoloured edges on this tour, she could have obtained a larger independent set of edges.

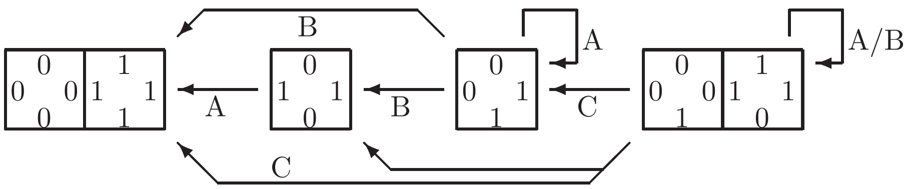
This contradiction shows that Max could never get to a vertex not incident with red edges, so that Olga always wins if she follows the above strategy. (Central Jury)

7. The task is guaranteed to succeed if and only if n is a power of 2.

Suppose n is not a power of 2. Then it has an odd prime factor p. Choose p evenly spaced barrels and make sure that the herrings inside are not all pointing the same way. Ignore all other barrels. At any point, let the herrings in r barrels are pointing up while the herrings in the other s barrels are pointing down. Since $r + s = p$ is odd, $r \neq s$.

We may assume that $r > s$. In order for Ali Baba to succeed, he must turn over all r barrels of the first kind or all s barrels of the second kind. A pagan god who is having fun with Ali Baba can spin the table so that if Ali Baba plans to turn over r barrels, the herring in at least one of them is pointing down; and if Ali Baba plans to turn over s barrels, the herring in all of them are pointing up. This way, Ali Baba will never be able to open the cave.

If $n = 2^k$ for some non-negative integer k, we will prove by induction on k that Ali Baba can open the cave. The case $k = 0$ is trivial as the cave opens automatically. The case $k = 1$ is easy. If the cave is not already open, turning one barrel over will do. For $k = 2$, let 0 or 1 indicate whether the herring is heads up or heads down.



The diagram above represents the four possible states the table may be in, as well as the transition between states by the following operations.

Operation **A:** Turn over any two opposite barrels.

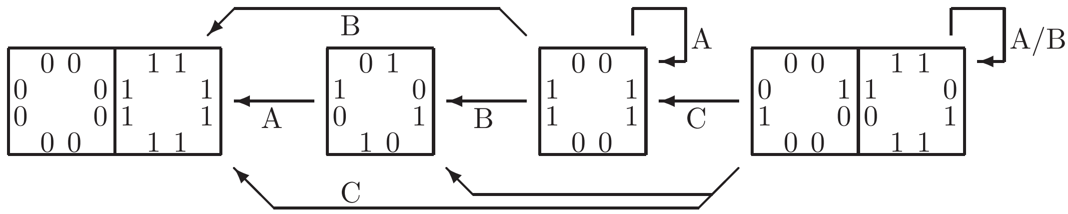
Operation **B:** Turn over any two adjacent barrels.

Operation **C:** Turn over any one barrel.

By performing the sequence **ABACABA**, the cave will open. The first state is called an *absorbing* state, in that once there, no further transition takes place as the cave will open immediately.

The second state becomes the first state upon the first operation A. The third state remains in place during the first operation A, but will become either the first state or the second state upon the first operation B. In the latter case, it will become the first state upon the second operation A. The fourth state remains in place during the first three operations, but will become any of the other three states upon the operation C. It will become the first state at the latest after three more operations.

The success of the case $k = 2$ paves the way for the case $k = 3$. The process is typical of the general inductive argument so that we give a detailed analysis. The idea is to treat each pair of diametrically opposite barrels as a single entity.



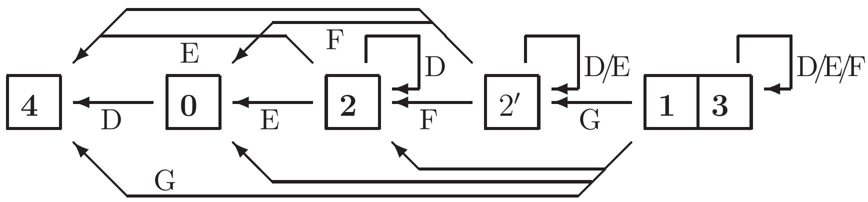
The above diagram, which is essentially copied from that for $k = 2$, is part of a much bigger state-transition diagram for $k = 3$. Here, all the states have the property that opposite pairs of barrels are all matching, that is, both are 0 or both are 1. The operations are modified from those in the case $k = 2$ as follows.

Operation **A:** Turn over every other pair of opposite barrels; in other words, turn over every other barrel.

Operation **B:** Turn over any two adjacent pairs of opposite barrels.

Operation **C:** Turn over any pair of opposite barrels.

By performing the sequence **ABACABA**, the cave will open. These states together form an expanded absorbing state in the overall diagram below.



Here, the box marked m contains all states with m matching opposite pairs, where $0 \leq m \leq 4$. The box marked 4 is the expanded absorbing state mentioned above. The states with 2 matching pairs are classified according to whether these matching are alternating or adjacent. The former states are contained in the box marked 2 while the latter states are contained in the box marked $2'$.

We have four new operations.

Operation **D:** Turn over any 4 adjacent barrels.

Operation **E:** Turn over any 2 barrels separated by one other barrel.

Operation **F:** Turn over any two adjacent barrels.

Operation **G:** Turn over any barrel.

Let **X** denote the sequence ABACABA. Then the sequence for the case $k = 3$ is

XDXEXDXFXDXEXDXGXDXEXDXFXDXEXDX.

We keep repeating X to clear any state that has entered the box marked 4, to prevent them from returning to another box. Whatever the state the table is in, the cave will open by the end of this sequence.

The general procedure is now clear. We treat each opposite pair as a single entity, thereby reducing to the preceding case. Then we moving progressively all states into the expanded absorbing state. Thus the task is possible whenever n is a power of 2.

(Hsin-Po Wang)