International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2011.**

1. Several guests at a round table are eating from a basket containing 2011 berries. Going in clockwise direction, each guest has eaten either twice as many berries as or six fewer berries than the next guest. Prove that not all the berries have been eaten.

2. Peter buys a lottery ticket on which he enters an n-digit number, none of the digits being 0. On the draw date, the lottery adminisrators will reveal an $n \times n$ table, each cell containing one of the digits from 1 to 9. A ticket wins a prize if it does not match any row or column of this table, read in either direction. Peter wants to bribe the admniistrators to reveal the digits on some cells chosen by Peter, so that Peter can guarantee to have a winning ticket. What is the minimum number of digits Peter has to know?

3. In a convex quadrilateral $ABCD$, $AB = 10$, $BC = 14$, $CD = 11$ and $DA = 5$. Determine the angle between its diagonals.

4. Positive integers $a < b < c$ are such that $b + a$ is a multiple of $b - a$ and $c + b$ is a multiple of $c - b$. If a is a 2011-digit number and b is a 2012-digit number, exactly how many digits does c have?

5. In the plane are 10 lines in general position, which means that no 2 are parallel and no 3 are concurrent. Where 2 lines intersect, we measure the smaller of the two angles formed between them. What is the maximum value of the sum of the measures of these 45 angles?

Note: The problems are worth 3, 4, 4, 4 and 5 points respectively.

Solution to Senior O-Level Fall 2011

1. It is not possible for each guest to eat six fewer berries than the next guest. Hence one of them has to eat twice as many, and therefore an even number of berries. Going now in the counter-clockwise direction, the next guest eats either twice as many as or six fewer than the preceding guest. It follows that every guest has eaten an even number of berries. Since 2011 is odd, not all the berries have been eaten.

2. The minimum number is n. If Peter knows at most $n-1$ of the digits, he will not know any digit on one of the rows, and his ticket may match that row. On the other hand, if Peter knows every digit on a diagonal, he can guarantee to have a winning ticket. Let the digits on this diagonal be $d_1, d_2, \ldots, d_n$. Peter can enter the digits $t_1, t_2, \ldots, t_n$ on his ticket such that neither t_k nor t_{n+1-k} matches d_k or d_{n+1-k} for any k, $1 \leq k \leq n$. Then his ticket cannot match the k-th row or the k-th column for any k in either direction.

3. Let AC and BD intersect at O. Suppose the diagonals are not perpendicular to each other. By symmetry, we may assume that $\angle AOB = \angle COD < 90°$. Then

$$(OA^2 + OB^2) + (OC^2 + OD^2) > AB^2 + CD^2 = 221$$

while

$$(OD^2 + OA^2) + (OB^2 + OC^2) < DA^2 + BC^2 = 221.$$

This is a contradiction. Hence both angles between the diagonals are $90°$.

4. Since $c > b$, c has at least 2012 digits. We have $b + a = k(b - a)$ for some integer $k > 1$. Hence $a(k+1) = b(k-1)$, so that $\frac{b}{a} = \frac{k+1}{k-1} = 1 + \frac{2}{k-1} \leq 3$, with equality if and only if $k = 2$. Similarly, $\frac{c}{b} \leq 3$, so that $\frac{c}{a} = \frac{c}{b} \cdot \frac{b}{a} \leq 9$. Hence $c < 10a$. Since a has 2011 digits, c has at most 2012 digits. Since $b < c$ and b has 2012 digits, it follows that c has exactly 2012 digits.

5. The answer is $25 \cdot 90\circ = 2250\circ$.

   Example. Consider 5 pairs of mutually perpendicular lines. Then the sum of two angles that any line forms with two lines of another pair equals $90°$. The angle inside a pair also equals $90°$. In total, we have $5 \cdot 90°$ for each line. Summing up the sums for all lines and dividing by 2 (since each angle was counted twice), we get $10 \cdot 5 \cdot 90°/2$. *Estimate.* We may parallel shift the 10 lines to make them concur not changing the angles between them. Now form 5 pairs from these lines so that two lines of any pair are separated by 4 other lines. Paint a pair blue, and another pair red. Red lines divide each angle between blue lines into two parts such that the sum of four parts is $180°$. These parts correspond to pairs formed by a blue and a red line, the least angles in these pairs don't exceed these parts, hence the sum of the blue-red angles is not greater than $180°$. We have 10 pairs of pairs, hence the sum of angles between lines different pairs doesn't exceed $10 \cdot 180°$. Each angle between lines of the same pair doesn't exceed $90°$. This implies the answer.